The Astroid and the Bicorn

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Abstract

There are many functions that are explored and researched by mathematicians. The astroid and the bicorn are examples of functions that were discovered to invent new ideas. The astroid was discovered by Roemer in 1674, and the bicorn was discovered by Sylvester in 1864.

Introduction

While studying these functions, there are multiple techniques that can be used to effectively analyze them. For example, one can find the perimeter of an object or area under the curve by utilizing calculus techniques. Other instances where calculus is useful is when finding the volume or the surface area of a function. This project explores these areas of calculus as it relates to the astroid and bicorn. For the bicorn, we explored arc length and area between curves, and for the astroid, we explored the area, surface area, arc length, and volume. All of these methods require integration, and sometimes derivation.

Methods and Results

The Astroid

Surface Area:

$$S = \int_{t_1}^{t_2} 2\pi ds$$

$$ds = \sqrt{(\frac{dx}{dy})^2 + (\frac{dy}{dx})^2}$$

$$ds = \sqrt{9a^2 sin^2(t) cos^2(t)}$$

$$ds = 3\,\mathrm{asin}(t)\cos(t)$$

$$S = \int_0^{\frac{\pi}{2}} 2\pi a sin^3(t)(\,3\,\mathrm{asin}(t)\cos(t))\mathrm{dt}$$

$$S = \int_0^{\frac{\pi}{2}} 6\pi a^2 sin^4(t) cos\,(t)\mathrm{dt}$$

$$S = 6\pi a^2 [(\frac{sin^5(t)}{5})]\,\frac{\pi}{2}\,0$$

$$S = \frac{6\pi a^2}{5}$$

However, this is only for one half.

$$S = 2(\frac{6\pi a^2}{5})$$

$$S = \frac{12\pi a^2}{5}$$

Volume:

$$V = \int_0^a 2\pi y^2 dx$$

$$V = 2\pi \int_0^a y^2 dx$$

$$y = asin^3(\mathrm{t})$$

$$\frac{dx}{dx} = -3\mathrm{asin}(t)cos^2(t)$$

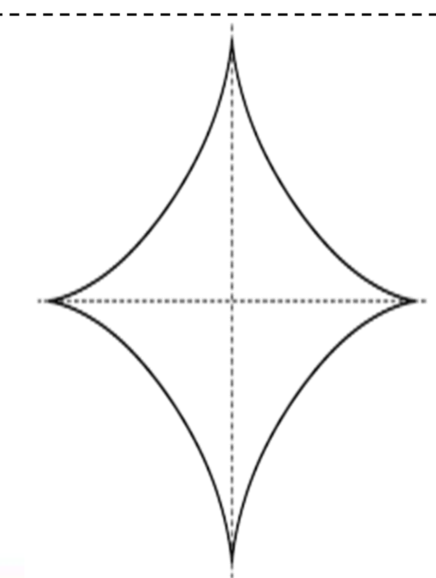
$$V = 2\pi \int_{\frac{\pi}{2}}^0 -a^2 sin^6(t) 3\,\mathrm{asin}(t)\, cos^2(t)dt$$

$$V = 6\pi a^3 \int_0^{\frac{\pi}{2}} sin^7(t)cos^2(t)dt$$

$$V = 6\pi a^3 \left(\frac{16}{315}\right)$$

$$V = \frac{96\pi a^3}{315}$$

$$V = \frac{32\pi a^3}{105}$$



The Bicorn

Arc Length

$$\mathrm{L} = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Bicorn is given by:

$$y = \frac{2-2x^2+\sqrt{1-3x^2+3x^4-x^6}}{3+x^2} \text{ (Top Curve)}$$

$$y = \frac{2-2x^2-\sqrt{1-3x^2+3x^4-x^6}}{3+x^2} \text{ (Bottom Curve)}$$

We find $\frac{dy}{dx}$ using quotient rule:

$$\frac{dy}{dx} = = \frac{3x\sqrt{1-x^2}-4x}{(x^2+3)} - \frac{2x(-1(1-x^2)^{\frac{3}{2}}-2x^2+2)}{(x^2+3)^2}$$

We plug into arc length formula:

$$L_b = \int_{-1}^1 \sqrt{1 + \left(\frac{3x\sqrt{1-x^2}-4x}{(x^2+3)} - \frac{2x\left(-1(1-x^2)^{\frac{3}{2}}-2x^2+2\right)}{(x^2+3)^2}\right)^2}\, dx$$

This cannot be solved symbolically. Using Maple to integrate numerically we get $L_b \approx 2.181667891$. The formula for the arc length of the top is very similar. The arc length for the top curve is given by:

$$L_t = \int_{-1}^1 \sqrt{1 + \left(\frac{-3x\sqrt{1-x^2}-4x}{(x^2+3)} - \frac{2x\left(-1(1-x^2)^{\frac{3}{2}}-2x^2+2\right)}{(x^2+3)^2}\right)^2}\, dx$$

Once again, this cannot be integrated numerically. We use Maple to integrate and we get $L_t \approx 2.849805127$. Adding these together to get the final arc length we get $L_{total} \approx 5.031473018$

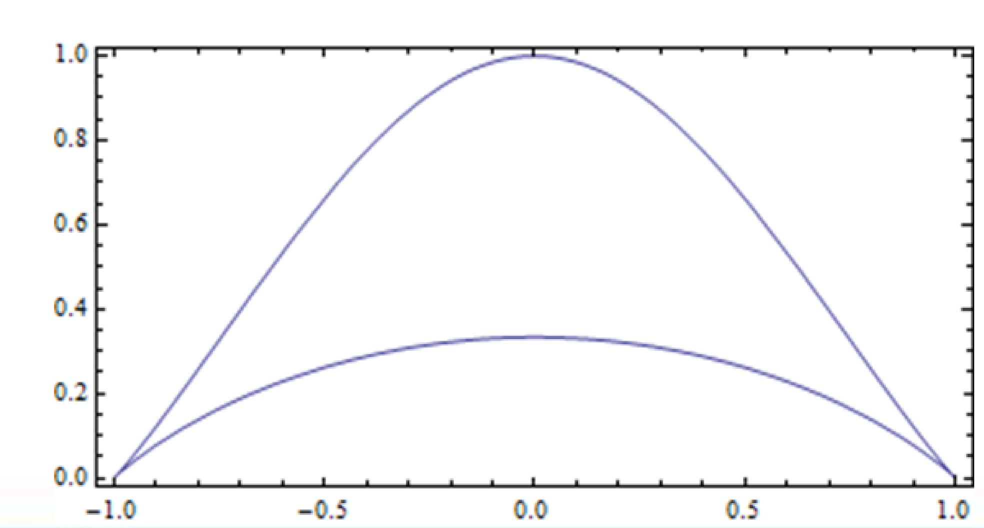
Area Between Curves

We must integrate both curves:

$$A_b = \int_{-1}^1 \frac{2-2x^2-\sqrt{1-3x^2+3x^4-x^6}}{3+x^2} dx$$

$$A_t = \int_{-1}^1 \frac{2-2x^2+\sqrt{1-3x^2+3x^4-x^6}}{3+x^2} dx$$

We use a combination of u-substitutions, trig substitutions, and partial fraction decomposition in order to integrate. We integrate both equations and subtract the area of the top from the area of the bottom. We get: $A_{total} \approx .6974160945$



Discussion

In our studies we found that it was much more difficult to analyze these curves than we previously thought. The curves look relatively simple yet the underlying math can be very complex. We did find limitations during our studies. One limitation was the limit of our math knowledge. With a more expanded knowledge we may have been able to use more efficient and accurate techniques to obtain our final results. Our results are important because they have applications in the mathematical community. The next steps of our research would be to expand our analysis on these two curves by using different techniques and by finding different characteristics of the curves.

References

"Astroid." *Astroid -- from Wolfram MathWorld*. N.p., n.d. Web. 05 Apr. 2017.

"Bicorn." *Bicorn -- from Wolfram MathWorld*. N.p., n.d. Web. 05 Apr. 2017.

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